Differential Geometry III: Riemannian Geometry

10.1 Let (S, g) be a Riemannian surface (i.e. $dim S = 2$). Suppose that, in polar coordinates (r, θ) around a point $p \in S$, the metric g takes the form

$$g = dr^2 + \big(f(r,\theta)\big)^2 d\theta^2$$

(recall that, as we showed in class, $\lim_{r\to 0} f(r,\theta) = 0$ and $\lim_{r\to 0} \partial_r f(r,\theta) = 1$).

(a) Show that the sectional curvature K of (S, g) satisfies at any point in this coordinate chart:

$$\frac{\partial^2 f}{\partial r^2} + Kf = 0.$$

(b) Derive an expression in polar coordinates for any metric of constant sectional curvature in dimension 2.

(c) Show that any two Riemannian surfaces with constant sectional curvature of the same value are locally isometric. Are they also globally isometric?

10.2 (a) Let $F : (\mathcal{M}, g) \to (\mathcal{N}, h)$ be an isometry. Show that, for any $X, Y, Z, W \in \Gamma(\mathcal{M})$ and any $p \in \mathcal{M}$,

$$R_h(F_*X, F_*Y, F_*Z, F_*W)|_{F(p)} = R_g(X, Y, Z, W)|_p,$$

where R_g, R_h are the Riemann curvature tensors associated to g, h, respectively, and $F_*(V) \doteq dF(V)$. *Hint: Use the fact that, for any such isometry F, $\nabla^{(h)}_{F_*X}(F_*Y) = F_*\big(\nabla^{(g)}_X Y\big)$.*

(b) Let $(\mathcal{M}, g)$ have the property that, for any $p, q \in \mathcal{M}$, any non-collinear $V_1, V_2 \in T_p\mathcal{M}$ and non-collinear $W_1, W_2 \in T_q\mathcal{M}$, there exists an isometry $F : \mathcal{M} \to \mathcal{M}$ such that $F(p) = q$ and the plane spanned by $\{F_*V_1, F_*V_2\}$ is the same as for $\{W_1, W_2\}$. Show that the sectional curvature is constant on $\mathcal{M}$, i.e. that for any $p \in \mathcal{M}$ and any $X, Y \in T_p\mathcal{M}$ which are not collinear, $K(X, Y)|_p$ has the same value K. A Riemannian manifold with the last property is called a **space form**. Show that the Riemann curvature tensor satisfies in this case:

$$R(X, Y, Z, W) = K \cdot \big(g(X, Z)g(Y, W) - g(X, W)g(Y, Z)\big).$$

Remark. For $n \leqslant 3$, every isotropic Riemannian manifold is a space form; this is not true for $n \geqslant 4$.

10.2 (a) Compute the sectional curvature of the hyperbolic plane $(\mathbb{H}^2, g_{\mathbb{H}})$. *Hint: Use the expression of $g_{\mathbb{H}}$ in polar coordinates.)*

(b) Compute the Riemann curvature tensor, Ricci tensor and sectional curvature tensor of $(\mathbb{S}^n, g_{\mathbb{S}^n})$. *Hint: You can do the computations directly in one of the coordinate expressions of $g_{\mathbb{S}^n}$ that we've seen in the exercises, or note that $(\mathbb{S}^n, g_{\mathbb{S}^n})$ is a space form.*

10.3 Let $f:\mathbb{R}^n\to\mathbb{R}$ be a smooth function. Consider the submanifold $\mathcal{M}_f$ of $\mathbb{R}^{n+1}=\mathbb{R}\times\mathbb{R}^n$ which is the graph of f, i.e.

$$\mathcal{M}_f=\big\{(t,x)\in\mathbb{R}\times\mathbb{R}^n:\, t=f(x)\big\}.$$

Compute the second fundamental form and the Riemann curvature tensor of the induced metric on $\mathcal{M}_f$. (*Hint: You might want to use the Gauss equation for the latter calculation.*)

10.4 Let $(\mathcal{M}^n,g)$ be a smooth Riemannian manifold and let p be a point on $\mathcal{M}$. For any given $0<\bar{r}<\iota(p)$, let us consider the open neighborhood $\mathcal{U}=\exp_p\Big(\big\{v:\|v\|<\bar{r}\big\}\Big)$ of p. Recall that $\mathcal{U}\setminus\{p\}$ is prametrized by the polar coordinates $(r,\omega)\in(0,\bar{r})\times\mathbb{S}^{n-1}$, where $r(\cdot)=dist_g(\cdot)$. Recall also that, in any local coordinate chart $(x^1,\dots,x^{n-1})$ on $\mathbb{S}^{n-1}$, the metric g in the $(r,x^1,\dots,x^{n-1})$ coordinate system takes the form

$$g=dr^2+r^2\bar{g}_{ij}[r]dx^idx^j,$$

where $\bar{g}_{ij}[r]\xrightarrow{r\to0}(g_{\mathbb{S}^{n-1}})_{ij}$ and $\partial_r\bar{g}_{ij}[r]\xrightarrow{r\to0}0$ (with $g_{\mathbb{S}^{n-1}}$ denoting the standard round metric on the unit sphere).

(a) Show that

$$\partial_r\big(r^2\bar{g}_{ij}[r]\big)=-2b_{ij}[r],$$

where $b[\rho]$ is the scalar second fundamental form of the hypersurface $S_\rho=\{r=\rho\}$ with respect to the coorientation determined by $\mathrm{grad}r$. *(Hint: Use Exercise 11.1.b.)*

(b) Show that

$$\partial_r b_{ij}[r]+r^{-2}\bar{g}^{ab}[r]\cdot b_{ia}[r]\cdot b_{jb}[r]=R_{rirj},$$

where R is the Riemann curvature tensor of g.

* (c) Show that if $R\equiv0$, then $\bar{g}_{ij}[r]=(g_{\mathbb{S}^{n-1}})_{ij}$ for all $r\in(0,\bar{r})$. Deduce, in this case, that g is isometric to the flat metric g_E. *(Hint: Show that, in this case, the tensor $M^i_j[r]=r^{-2}\bar{g}^{ia}[r]\cdot b_{jb}[r]$ on S_r satisfies, with respect to r, the matrix Ricatti ODE $\partial_rM-M^2=0$. What is $\lim_{r\to0}M$?)*

10.5 Let $(\mathcal{M}_1,g_1)$ and $(\mathcal{M}_2,g_2)$ be two Riemannian manifolds and let $(\mathcal{M},g)=(\mathcal{M}_1\times\mathcal{M}_2,g_1\oplus g_2$ be their Riemannian product; the metric $g_1\oplus g_2$ is defined so that, for any $p=(p_1,p_2)\in\mathcal{M}_1\times\mathcal{M}_2$ and any $X,Y\in T_p\mathcal{M}\simeq T_{p_1}\mathcal{M}_1\oplus T_{p_2}\mathcal{M}_2$, if $X=X_1+X_2$ and $Y=Y_1+Y_2$ is their corresponding decomposition into tangent vectors tangential to $\mathcal{M}_1\times\{p_2\}$ and $\{p_1\}\times\mathcal{M}_2$ then

$$g(X,Y)=g_1(X_1,Y_1)+g_2(X_2,Y_2)$$

(in other words, $\mathcal{M}_1\times\{p_2\}$ and $\{p_1\}\times\mathcal{M}_2$ intersect orthogonally and $\mathcal{M}_1\to\mathcal{M}_1\times\{p_2\}$ and $\mathcal{M}_2\to\{p_1\}\times\mathcal{M}_2$ are isometric embeddings).

(a) Compute the Riemann curvature tensor R of $(\mathcal{M},g)$ in terms of the Riemann curvature tensors R_i of $(\mathcal{M}_i,g_i)$, $i=1,2$.

(b) Show that the sectional curvature of $(\mathcal{M}, g)$ cannot be strictly positive or strictly negative for all tangent 2-planes.

(*c) Show that there exists a surface in $(\mathbb{S}^2 \times \mathbb{S}^2, g_{\mathbb{S}^2} \oplus g_{\mathbb{S}^2})$ which is totally geodesic (i.e. has vanishing second fundamental form) and is isometric to the flat torus $(\mathbb{T}^2, g_E)$.